Last update: 2:45pm, Apr 11.

Math 565: Functional Analysis HOMEWORK 6 **Due: Apr 12, 23:59**

0. [*Optional*] Let X be a real or complex vector space with a norm $\|\cdot\|$ satisfying the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$. Follow the outline below to prove that $\|\cdot\|$ is induced by an inner product, namely, the one defined by the polarization identity.

Viewing X as a real vector space, put

$$\langle x, y\rangle_{\mathbb{R}} := \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big).$$

(a) Prove $\langle x+z, y\rangle_{\mathbb{R}} + \langle x-z, y\rangle_{\mathbb{R}} = 2\langle x, y\rangle_{\mathbb{R}}$ for all $x, y, z \in X$. In particular, $\langle 2x, y\rangle_{\mathbb{R}} = 2\langle x, y\rangle_{\mathbb{R}}$.

HINT: Expand the left-hand side using the definition of $\langle\cdot,\cdot\rangle$, then apply the parallelogram law twice.

(b) Prove additivity in the first variable: for all $a, b, y \in X$

$$\langle a+b, y\rangle_{\mathbb{R}} = \langle a, y\rangle_{\mathbb{R}} + \langle b, y\rangle_{\mathbb{R}}.$$

Hint: Apply the previous part with $x := \frac{1}{2}(a+b)$ and $z := \frac{1}{2}(a-b)$.

(c) Deduce that $\langle nx, y\rangle_{\mathbb{R}} = n\langle x, y\rangle_{\mathbb{R}}$ for all integers $n \in \mathbb{Z}$, and then that $\langle qx, y\rangle_{\mathbb{R}} = q\langle x, y\rangle_{\mathbb{R}}$ for all rationals $q \in \mathbb{Q}$. Conclude by continuity that $\langle rx, y\rangle_{\mathbb{R}} = r\langle x, y\rangle_{\mathbb{R}}$ for all reals $r \in \mathbb{R}$.

Thus $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is a real inner product such that $\|x\| = \sqrt{\langle x, x\rangle_{\mathbb{R}}}$ for all $x \in X$. Now if X is a complex vector space, put

$$\langle x, y\rangle := \langle x, y\rangle_{\mathbb{R}} + i\langle x, iy\rangle_{\mathbb{R}},$$

so $\langle\cdot,\cdot\rangle$ is still additive in the first variable.

(d) Prove that $\langle ix, y\rangle = i\langle x, y\rangle$ and deduce $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{C}$.

(e) Observe that $\langle y, x\rangle = \overline{\langle x, y\rangle}$ and that still $\langle x, x\rangle = \|x\|^2$ to conclude that $\langle\cdot,\cdot\rangle$ is a complex inner product inducing the norm of X.

1. Let I be a (potentially uncountable) set.

(a) Prove that every function $f \in \ell^2(I)$ has countable support $\operatorname{supp} f := \{i \in I : f(i) \neq 0\}$. Deduce that a function $g : I \to \mathbb{C}$ is in $\ell^2(I)$ if and only if there is a constant $C > 0$ such that for each finite $I_0 \subseteq I$, we have $\sum_{i \in I_0} |g(i)|^2 < C$.

(b) Let H be a Hilbert space and $\{u_i\}_{i \in I} \subseteq H$ be an orthonormal family. Prove that for each $f : I \to \mathbb{C}$, the series $\sum_{i \in I} f(i)u_i$ converges in H if and only if $f \in \ell^2(I)$.

2. Let H be an infinite-dimensional Hilbert space. Prove:

(a) Every orthonormal sequence in H converges weakly to 0.

(b) The unit sphere $S = \{x \in H : \|x\| = 1\}$ is weakly dense in the closed unit ball $\overline{B}_1 = \{x \in H : \|x\| \leqslant 1\}$. In fact, every $x \in \overline{B}_1$ is the weak limit of a sequence in S.

3. Let H be a separable infinite-dimensional Hilbert space with orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$.

(a) Define $L \in B(H)$ by $L\Big(\sum_{n\geqslant 0} a_n u_n\Big) := \sum_{n\geqslant 1} a_n u_{n-1}$. Then $L^k \to 0$ in the strong operator topology but not in the norm topology.

(b) Define $R \in B(H)$ by $R\Big(\sum_{n\geqslant 0} a_n u_n\Big) := \sum_{n\geqslant 0} a_n u_{n+1}$. Then $R^k \to 0$ in the weak operator topology but not in the strong operator topology.

4. **Mean Ergodic Theorem**. Let U be a unitary operator on the Hilbert space H and let $M := \{x : Ux = x\}$ be the subspace of U-invariant vectors, so M is closed (why?). Let

$$S_n := \frac{1}{n+1}\sum_{j=0}^{n} U^j.$$

Prove that $S_n \to \operatorname{proj}_M$ in the strong operator topology.

HINT: Let $T := \operatorname{id}_H - U$ and observe that $M = \ker T^*$, so $H = M + \overline{\operatorname{im} T}$ and work with each orthogonal subspace separately.

5. Let X be a Banach space and H be a Hilbert space. Let $T \in B_c(X, H)$ be a compact transformation. Prove:

(a) $\overline{T(X)}$ is a separable subspace of H.

(b) T is a norm-limit of finite-rank operators.

HINT: One way to prove it is to use a countable orthonormal basis of $\overline{T(X)}$. A slightly faster way is to let $F \subseteq T(\overline{B}_1^X)$ be a finite set that is ε-dense in $T(\overline{B}_1^X)$ (i.e. every $y \in T(\overline{B}_1^X)$ is within distance $\leqslant \varepsilon$ from F), and consider the orthogonal projection onto $\operatorname{span} F$.

REMARK: This statement is false in general when H is replaced with a Banach space Y. The problem with the proof for Y is that projections to finite dimensional subspaces (i.e. maps to the nearest point), although they exist, may not be unique, and even when they are unique this map may not be linear.

Banach spaces into which every compact transformation is a norm-limit of finite-rank operators are exactly those with the so-called *approximation property*, which is pretty much exactly what is needed for the second hinted proof above to go through.

(c) Deduce that the adjoint T^* of a compact operator $T \in B_c(H)$ is compact.

6. Let (X, μ) be a σ-finite measure space. Prove that every Hilbert–Schmidt operator T on $L^2(X, \mu)$ is a kernel operator T_K for some $K \in L^2(X \times X, \mu \times \mu)$. Thus, the map $K \mapsto T_K : L^2(X \times X, \mu \times \mu) \to B_2(H)$ is an isometric isomorphism, where $B_2(H)$ is the space of all Hilbert–Schmidt operators equipped with the Hilbert–Schmidt norm.

HINT: Let $\{e_i\}_{i\in I}$ be an orthonormal basis for H and put $K(x, y) := \sum_{i\in I}(Te_i)(x)\overline{e_i(y)}$. To show convergence of this series, note that the summands $(x, y) \mapsto (Te_i)(x)\overline{e_i(y)}$ are pairwise orthogonal vectors in $L^2(X \times X, \mu \times \mu)$.

REMARK: Note that $B_2(H)$ is itself a Hilbert space with the inner product $\langle T, S\rangle := \sum_{i,j\in I} T_{ij}\overline{S_{i,j}}$ which induces the Hilbert–Schmidt norm. Thus, the map $K \mapsto T_K$ is, in fact, a unitary isomorphism of Hilbert spaces.

Definition. Let X be a Banach space and let $T \in B(X)$.

- The **resolvent set** of T is $\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible in } B(X)\}$.
- The **spectrum** of T is $\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(X)\}$. The **spectral radius** of T is $\|T\|_\sigma := \sup\{|\lambda| : \lambda \in \sigma(T)\}$.
- The **point spectrum** of T is $\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}$, i.e. the set of eigenvalues of T.

7. Let X be a Banach space and $T \in B(X)$. Let I denote the identity on X. Prove:

(a) $\|T\|_\sigma \leqslant \|T\|$. In fact, if $|\lambda| > \|T\|$, then $(T - \lambda I)^{-1} = -\sum_{n\in\mathbb{N}} \lambda^{-(n+1)} T^n$.

HINT: Question 4(b) of Homework 2.

(b) The resolvent $\rho(T)$ is open. Hence the spectrum $\sigma(T)$ is compact.

HINT: Question 4(c) of Homework 2.

(c) If $\sigma(T) = \emptyset$ then the map $f : \lambda \mapsto (T - \lambda I)^{-1} : \mathbb{C} \to B(X)$ is bounded.

HINT: f is continuous, so it suffices to show that f is bounded on $\{\lambda \in \mathbb{C} : |\lambda| > 2\|T\|\}$. Estimate $\|f(\lambda)\|$ from above using part (a) and Question 4(a) of Homework 2.

Definition. Let (X, μ) be a measure space. For each measurable $f : X \to \mathbb{C}$, its **essential range** is the set

$$\operatorname{essran} f := \left\{\lambda \in \mathbb{C} : \mu(f^{-1}(U)) > 0 \text{ for each open } U \ni \lambda\right\}.$$

Observe that $\operatorname{essran} f$ is closed by definition and that $\|f\|_\infty = \sup(\operatorname{essran}|f|)$.

8. Let (X, μ) be a **semifinite measure space**, i.e. every set of positive measure has a subset of *finite* positive measure. Let $\lambda \in L^\infty(X, \mu)$. Let $M_\lambda : L^2(X, \mu) \to L^2(X, \mu)$ be defined by $f \mapsto \lambda f$. Prove:

(a) M_λ is a bounded linear operator with $\|M_\lambda\| = \|\lambda\|_\infty$. Operators of this form are called **multiplication operators**.

REMARK: In Example (a) at the very end of Lecture 5, we proved a related statement, but not this statement, since in that example $q < \infty$.

(b) M_λ is invertible $\iff \lambda^{-1} \in L^\infty(X, \mu) \iff 0 \notin \operatorname{essran} \lambda$.

(c) $\sigma(M_\lambda) = \operatorname{essran} \lambda$.